Dynamical Borel-Cantelli lemmas for Gibbs measures

N. Chernov* and D. Kleinbock†

To appear in Israel J. Math.

Abstract

Let $T : X \mapsto X$ be a deterministic dynamical system preserving a probability measure μ. A dynamical Borel-Cantelli lemma asserts that for certain sequences of subsets $A_n \subset X$ and μ-almost every point $x \in X$ the inclusion $T^n x \in A_n$ holds for infinitely many n. We discuss here systems which are either symbolic (topological) Markov chain or Anosov diffeomorphisms preserving Gibbs measures. We find sufficient conditions on sequences of cylinders and rectangles, respectively, that ensure the dynamical Borel-Cantelli lemma.

1 Introduction

Let $T : X \mapsto X$ be a transformation preserving a probability measure μ. We use notation $\mu(f) := \int f \, d\mu$ for integrable functions f on X.

Let $A_n \subset X$ be a sequence of measurable sets. Put $B_n = T^{-n} A_n$ and consider the set

$$\limsup_n B_n := \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} B_n$$

of points which belong to infinitely many B_n. A classical Borel-Cantelli lemma in probability theory states:

Lemma 1.1 (Borel-Cantelli) (i) *If $\sum \mu(B_n) < \infty$, then $\mu(\limsup_n B_n) = 0$, i.e. almost every point $x \in X$ belongs to finitely many B_n.*
(ii) *If $\sum \mu(B_n) = \infty$ and B_n are independent, then $\mu(\limsup_n B_n) = 1$, i.e. almost every point $x \in X$ belongs to infinitely many B_n.*

In terms of the transformation T, the lemma can be restated as follows.

Lemma 1.2 (i) *If $\sum \mu(A_n) < \infty$, then for almost every point $x \in X$ there are only finitely many n such that $T^n x \in A_n$.*
(ii) *If $\sum \mu(A_n) = \infty$ and $T^{-n} A_n$ are independent, then for almost every point $x \in X$ there are infinitely many n such that $T^n x \in A_n$.*

[0*] Partially supported by NSF grant DMS-9732728.
[0†] Partially supported by NSF grant DMS-9704489.

The second part of the lemma has a limited value for deterministic dynamical systems, since one rarely works with purely independent sets. This paper is devoted to extensions of the second part of the lemma to certain dynamical systems – Anosov diffeomorphisms and topological Markov chains.

Below we always assume that $\sum_n \mu(A_n) = \infty$.

Definition. A sequence of subsets $A_n \subset X$ is called a *Borel-Cantelli* (BC) sequence if for μ-a.e. $x \in X$ there are infinitely many n such that $T^n x \in A_n$.

Let

$$\chi_n(x) := \chi_{T^{-n}A_n}(x)$$

be the indicator of the set $B_n = T^{-n}A_n$. We set

$$S_N(x) := \sum_{n=1}^{N} \chi_n(x)$$

and

$$E'_N := \mu(S_N) = \sum_{n=1}^{N} \mu(A_n) .$$

Definition. A sequence of subsets $A_n \subset X$ is said to be a *strongly Borel-Cantelli* (sBC) sequence if for μ-a.e. $x \in X$ we have $S_N(x)/E_N \to 1$ as $N \to \infty$.

A stronger version of the classical Borel-Cantelli lemma is known, see Theorem 6.6 in [6]:

Lemma 1.3 *If $\sum \mu(B_n) = \infty$ and the events B_n are independent, then $S_N(x)/E_N \to 1$ almost surely as $N \to \infty$. Moreover, the independence requirement can be relaxed to the pairwise independence, i.e. it is enough to require $\mu(B_m \cap B_n) = \mu(B_m)\mu(B_n)$ for $m \neq n$.*

In particular, if $B_n = T^{-n}A_n$ are pairwise independent, then the sequence $\{A_n\}$ is an sBC sequence.

Consider the quantity

$$R_{mn} := \mu(B_m \cap B_n) - \mu(B_m)\mu(B_n) = \mu(T^{-m}A_m \cap T^{-n}A_n) - \mu(A_m)\mu(A_n)$$

which characterizes the dependence of B_m and B_n.

A sufficient condition for $\{A_n\}$ to be an sBC sequence, in terms of R_{mn}, was first found by W. Schmidt, see a proof by Sprindžuk [14], in the context of Diophantine approximations. It was recently adapted to dynamical systems by D. Kleinbock and G. Margulis [9]:

(SP) Assume that

$$\exists C > 0 : \qquad \sum_{m,n=M}^{N} R_{mn} \leq C \cdot \sum_{n=M}^{N} \mu(A_n)$$

for all $N \geq M \geq 1$.

Theorem 1.4 (**[14]**, Chapter I, Lemma 10, or **[9]**, Lemma 2.6) *If the sequence* $\{A_n\}$ *satisfies* (SP)*, then it is an sBC sequence; moreover, for a.e.* $x \in X$ *one has*

$$S_N = E_N + O\Big(E_N^{1/2} \log^{3/2+\varepsilon} E_N\Big). \tag{1.1}$$

W. Philipp was first to derive the asymptotics (1.1) in the context of dynamical system, and he called it a quantitative Borel-Cantelli lemma [11].

Note that there exist remarkable characterizations of some ergodic properties of dynamical systems in terms of BC and sBC sequences. We summarize these in the following

Proposition 1.5 *Let* T *be a measure preserving transformation of a probability space* (X, μ)*. Then:*
(i) T *is ergodic* $\iff$ *every constant sequence* $A_n \equiv A$*,* $\mu(A) > 0$*, is BC* $\iff$ *every such sequence is sBC, i.e.* $S_N/E_N \to 1$ μ*-almost everywhere;*
(ii) T *is weakly mixing* $\iff$ *every sequence* $\{A_n\}$ *that only contains finitely many distinct sets, none of them of measure zero, is BC* $\iff$ *for every such sequence one has* $S_N/E_N \to 1$ *in the* L^2 *metric, i.e.* $\mu(S_N/E_N - 1)^2 \to 0$*;*
(iii) T *is lightly mixing*[1] $\iff$ *every sequence that only contains finitely many distinct sets, possibly of measure zero, is BC.*

See Section 3 for the proof. Note that in part (ii), the first equivalence was proved by Y. Guivarc'h and A. Raugi (private communication); our proof is slightly different. Part (iii) was pointed out to us by A. del Junco.

Note also that there exist no measure-preserving system such that every sequence $\{A_n\}$ that only contains two distinct sets, one of positive measure and the other of measure zero, is sBC. This follows from a result of U. Krengel [10]. On the other hand, if μ has K property, then any sequence that only contains finitely many sets, none of them of measure zero, is sBC (J.-P. Conze, private communication).

It is important to mention that for any (nontrivial) measure-preserving system (X, μ, T) there are sequences of subsets of X (with divergent sum of measures) which are not BC. More precisely, the following is true:

Proposition 1.6 *Let* (X, μ) *be a probability space. If* μ *is nontrivial (that is, there are sets with measure strictly between* 0 *and* 1*), then for any* μ*-preserving transformation* T *of* X *there exists a sequence* $\{A_n\}$ *of measurable subsets of* X *with* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *which is not BC. Furthermore, if* μ *is non-atomic, then for any* μ*-preserving transformation* T *of* X *there exists a sequence* $\{A_n\}$ *of measurable subsets of* X *with* $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ *such that for a.e.* $x \in X$ *there are at most finitely many* n *for which* $T^n x \in A_n$*.*

[1] T is said to be lightly mixing (see [7]) if for every two sets A, B of positive measure one has $\mu(T^{-n}A \cap B) > 0$ for large enough n; this condition lies strictly between mixing and weak mixing.

See the end of Section 3 for the proof. With a little extra work, one can always find a non BC sequence of sets that are nested: $A_1 \supset A_2 \supset \cdots$. We omit the proof.

Observe that a non-BC sequence can be easily constructed when T is invertible: one can simply take $A_n = T^n A$, where $0 < \mu(A) < 1$. Therefore to prove the BC or sBC property for certain classes of sequences it is necessary to impose certain restrictions on the sets A_n, which, roughly speaking, guarantee that the sets B_m and B_n become nearly independent for large $|m-n|$.

The first Borel-Cantelli lemma for deterministic dynamical systems was proved in 1969 by W. Philipp:

Theorem 1.7 ([11]) *Assume that $T(x) = \beta x$ (mod 1) with $\beta > 1$, or $T(x) = \{1/x\}$ (the Gauss transformation) and μ is the unique T-invariant smooth measure on $[0,1]$. Then any sequence $\{A_n\}$ of subintervals (with divergent sum of measures) is an sBC sequence, and (1.1) holds.*

In particular, one can take any $x_0 \in (0,1)$ and consider what could be called "a target shrinking to x_0" (terminology borrowed from [8]), i.e. a sequence of intervals $A_n = (x_0 - r_n, x_0 + r_n)$ with $r_n \to 0$. Then almost all orbits $\{T^n x\}$ get into infinitely many such intervals whenever r_n decays slowly enough. This can be thought of as a quantitative strengthening of density of almost all orbits (cf. the paper [1] for a similar approach to the rate of recurrence).

More generally, if X is a metric space (e.g. a Riemannian manifold), one can try to prove that any sequence $\{A_n\}$ of balls in X is BC or sBC; as in the example above, this would imply that all points $x_0 \in X$ can be "well approximated" by orbit points $T^n x$ for almost all x. D. Dolgopyat recently proved the following:

Theorem 1.8 ([5]) *Let $T : X \mapsto X$ be an Anosov diffeomorphism with a smooth invariant probability measure μ. Then any sequence of round balls (with divergent sum of measures) is sBC.*

Another example of a dynamical Borel-Cantelli lemma is given in the paper [9], where the following theorem was essentially proved:

Theorem 1.9 ([9]) *Let G be a connected semisimple center-free Lie group without compact factors, Γ an irreducible lattice in G, μ the normalized Haar measure on G/Γ, g a partially hyperbolic element of G, and let T be the left shift $T(x) = gx$, $x \in G/\Gamma$. Let $\{A_n\}$ be a sequence of subsets of G/Γ with divergent sum of measures and "uniformly regular boundaries", namely, such that for some $\delta > 0$ and $0 < c < 1$ one has*

$$\mu(\delta\text{−neighborhood of } \partial A_n) \le c\mu(A_n) \quad \text{for all } n\,. \tag{1.2}$$

Then there exist positive C_1, C_2 such that for μ-a.e. $x \in G/\Gamma$ one has

$$C_1 \le \liminf_{N\to\infty} S_N(x)/E_N \le \limsup_{N\to\infty} S_N(x)/E_N \le C_2\,;$$

in particular, $\{A_n\}$ is a BC sequence.

It is shown in [9] that the above condition (1.2) is satisfied if G/Γ is not compact and the sets $\{A_n\}$ are *complements* of balls centered in a fixed point $x_0 \in G/\Gamma$. This way one gets a description of *growth* of almost all orbits $T^n x$ as follows: if a sequence R_n increases slowly enough, then for almost all x one has $\mathrm{dist}(x_0, T^n x) \geq R_n$ for infinitely many n. This has important applications to geometry and number theory.

When this paper was under preparation, we learned that J.-P. Conze and A. Raugi [4] proved a dynamical Borel-Cantelli lemma for certain Markov processes and one-sided topological Markov chains with Gibbs measures.

2 Statement of results

Our paper deals with Anosov diffeomorphisms and the corresponding symbolic systems – topological Markov chains.

Let $T : X \mapsto X$ be a transitive Anosov diffeomorphism. Let $\mathcal{R} = \{R_1, \dots, R_M\}$ be a finite Markov partition of X, and $\mathbf{A}$ the corresponding transition matrix of zeroes and ones. For definitions and basic facts on Markov partitions, see [2, 3].

The matrix $\mathbf{A}$ is transitive, i.e. $\mathbf{A}^K$ is completely positive for some $K \geq 1$. Let $\Sigma = \Sigma_{\mathbf{A}}$ be the *topological Markov chain* for $\mathbf{A}$, i.e. a set of doubly infinite sequences $\underline{\omega} = \{\omega_i\}_{i=-\infty}^{\infty} \in \{1, \dots, M\}^{\mathbb{Z}}$ defined by

$$\Sigma = \{\underline{\omega} \in \{1, \dots, M\}^{\mathbb{Z}} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}\} .$$

The set Σ equipped with the product topology is a compact space, and there is a left shift homeomorphism $\sigma : \Sigma \mapsto \Sigma$ defined by $(\sigma \underline{\omega})_i = \omega_{i+1}$. Let $\pi : \Sigma \mapsto X$ be the projection defined by

$$\pi(\underline{\omega}) = \cap_{i=-\infty}^{\infty} T^{-i} R_{\omega_i} .$$

Then π is a continuous surjection and $\pi \circ \sigma = T \circ \pi$. Fix an $a \in (0,1)$ and let d_a be a metric on Σ defined by $d_a(\underline{\omega}, \underline{\omega}') = a^n$ where $n = \max\{n : \omega_i = \omega_i', \ \forall |i| < n\}$. It is consistent with the product topology. The projection π is now Hölder continuous.

There are classes of *Gibbs measures* on both X and Σ defined by potential functions. For any Hölder continuous function $\psi : \Sigma \mapsto \mathbb{R}$ there is a unique σ-invariant Gibbs measure μ_ψ on Σ. For any Hölder continuous function $\varphi : X \mapsto \mathbb{R}$ there is a unique T-invariant Gibbs measure μ_φ on X. In the latter case, the function $\psi = \varphi \circ \pi$ is Hölder continuous on Σ, and the measure μ_ψ projects to μ_φ in the sense that $\pi : \Sigma \mapsto X$ is μ_ψ-almost everywhere one-to-one and $\pi_* \mu_\psi = \mu_\varphi$.

Gibbs measures include all practically interesting invariant measures on X and Σ, e.g. all smooth invariant measures on X, Sinai-Ruelle-Bowen (SRB) measures, measures of maximal entropy (i.e. Margulis measures on X and Parry measures on Σ) etc.

We first study topological Markov chains separately from Anosov diffeomorphisms. Let Σ be a topological Markov chain with a transitive matrix $\mathbf{A}$. Let μ be an arbitrary Gibbs measure defined by a Hölder continuous potential. Naturally interesting subsets of Σ are *cylinders*, which include all balls in the metric d_a.

A cylinder $C \subset \Sigma$ is obtained by fixing symbols on a finite interval $\Lambda = [n^-, n^+] \subset \mathbb{Z}$, i.e. for some $\omega_\Lambda \in \{1, \ldots, M\}^\Lambda$, $\omega_\Lambda = \{\omega_{n^-}, \ldots, \omega_{n^+}\}$, we set

$$C = C(\omega_\Lambda); = \{\underline{\omega}' \in \Sigma : \omega'_i = \omega_i \quad \text{for } \; n^- \le i \le n^+\} \tag{2.1}$$

Each cylinder is open and closed in Σ. We call n^- and n^+ the *left* and *right endpoints* of an interval Λ, respectively, and $(n^- + n^+)/2$ the *center* of Λ.

Note that not every sequence of cylinders is a BC sequence. For example, let $C_n = \sigma^n C$ for a fixed cylinder C. It is obviously not a BC sequence. Hence, we need some restrictions on cylinders to ensure quasi-independence of $\sigma^{m-n}C_n$ and C_m for large $|m - n|$.

Definition. We say that two intervals $[n_1^-, n_1^+]$ and $[n_2^-, n_2^+]$ are *D-nested* for $D \ge 0$ if either $[n_1^-, n_1^+] \subset [n_2^- - D, n_2^+ + D]$ or $[n_2^-, n_2^+] \subset [n_1^- - D, n_1^+ + D]$.

Theorem 2.1 *Let $\{C_n\}$ be a sequence of cylinders defined on intervals $\Lambda_n \subset \mathbb{Z}$. Let $D \ge 0$ be a constant. Assume that for all m, n the intervals Λ_m, Λ_n are D-nested. Then $\{C_n\}$ satisfies* (SP) *and hence, if in addition $\sum \mu(C_n) = \infty$, it is an sBC sequence and* (1.1) *holds.*

Examples.
1. Let the left endpoints of Λ_n lie in the interval $[0, D]$, then Λ_n are D-nested. We call such intervals Λ_n *D-aligned*. (Similarly one can talk about right endpoints.)
2. Let the centers of Λ_n lie in the interval $[-D/2, D/2]$, then Λ_n are D-nested. We call such intervals Λ_n *D-centered*. Note that cylinders defined on 0-centered intervals are precisely balls in Σ with respect to the metric d_a defined above. Therefore the "quantitative orbit density" phenomenon (see the discussion after Theorem 1.7) holds for Gibbs measures on topological Markov chains. Specifically, if one fixes $\underline{\omega}_0 \in \Sigma$ and considers "a target shrinking to $\underline{\omega}_0$", that is, a sequence of balls (or centered at $\underline{\omega}_0$, then μ-almost all orbits $\{\sigma^n \underline{\omega}\}$ get into infinitely many such balls whenever the sum of their measures diverges.

The following two theorems show that the assumptions of Theorem 2.1 cannot be easily relaxed. We need to introduce some terminology generalizing the two examples above. Let $\{l_n\}$ be a sequence of positive numbers. We say that a sequence $\{\Lambda_n\}$ of intervals is *$\{l_n\}$-centered* (resp. *$\{l_n\}$-aligned*) if the center (resp. the left endpoint) of each Λ_n belongs to $[-l_n/2, l_n/2]$ (resp., $[0, l_n]$).

Theorem 2.2 *Let $\{l_n\}$ be a sequence of natural numbers such that $l_n \to \infty$. Then there is a sequence of cylinders $\{C_n\}$ with divergent sum of measures which is defined on $\{l_n\}$-centered (or, alternatively, $\{l_n\}$-aligned) intervals $\Lambda_n \subset \mathbb{Z}$ and does not satisfy* (SP).

Theorem 2.3 *Let $\varepsilon > 0$. There is a sequence of cylinders $\{C_n\}$ with $\sum \mu(C_n) = \infty$ which is defined on $\{\varepsilon|\Lambda_n|\}$-centered (or, alternatively, $\{\varepsilon|\Lambda_n|\}$-aligned) intervals $\Lambda_n \subset \mathbb{Z}$ and is not a BC sequence. Moreover, for a.e. $\underline{\omega} \in \Sigma$ there are only finitely many n such that $\sigma^n \underline{\omega} \in C_n$.*

Theorems 2.2 and 2.3 show that it is not enough, even for the BC property, that the cylinders are 'relatively well' centered or aligned.

Remarks.
1. Suppose that each of the sets C_n is a union of at most k_n cylinders satisfying the nested condition. It is clear that the conclusion of Theorem 2.1 still holds when the sequence $\{k_n\}$ is bounded. On the other hand, Theorem 2.2 shows that a sequence of unions C_n of k_n 0-centered cylinders may not satisfy (SP) if $\{k_n\}$ is unbounded, while Theorem 2.3 shows that $\{C_n\}$ is not necessarily BC if k_n is of order n^a with some $a > 0$.
2. Consider a one-sided topological Markov chain $\sigma : \Sigma^+ \mapsto \Sigma^+$ defined on the space Σ^+ of one-sided sequences:

$$\Sigma^+ = \{\underline{\omega} \in \{1, \ldots, M\}^{\mathbb{Z}_+} : \ \mathbf{A}_{\omega_i \omega_{i+1}} = 1 \quad \forall i \in \mathbb{Z}_+\} \, ;$$

here $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Note that the shift σ preserves Σ^+ but is not invertible, every sequence $\underline{\omega}$ may have up to M preimages. One-sided topological Markov chains give symbolic representation for piecewise smooth expanding interval maps satisfying the Markov condition.

Theorems 2.1–2.3 apply to one-sided topologically mixing Markov chains without change. Note, however, that all the cylinders must be defined on intervals $\Lambda \subset \mathbb{Z}_+$. In particular, our theorems hold for cylinders defined on intervals that are D-aligned, $\{l_n\}$-aligned and $\{\varepsilon|\Lambda_n|\}$-aligned, respectively[2]. Consider the metric d_a^+ on Σ^+ given by $d_a^+(\underline{\omega}, \underline{\omega}') = a^n$ where $n = \max\{n : \omega_i = \omega_i', \ \forall i < n\}$. In this metric, balls are cylinders defined on 0-aligned intervals. Therefore the "quantitative orbit density" phenomenon, which follows from Theorem 1.7 if $\Sigma^+ = \{1, \ldots, M\}^{\mathbb{Z}_+}$ and μ is the product measure, is extended to hold for an arbitrary Gibbs measure on a one-sided topological Markov chain.

It is also worthwhile to mention that Theorem 2.3 gives examples of non-BC sequences of cylinders in the setting of one-sided shifts. In fact, the idea of the proof works for an arbitrary measure-preserving system and produces examples of non-BC sequences in the generality of Proposition 1.6.

Back to Anosov diffeomorphisms, the above theorems can be restated by replacing cylinders with their projections on the manifold X and the T-invariant measure $\pi_*\mu$ on X. The projection $\pi(C)$ of a cylinder $C = C(\omega_\Lambda)$ is a rectangle

$$\pi(C) = \cap_{i=n^-}^{n^+} T^{-i} R_{\omega_i} \tag{2.2}$$

in terms of of the formula (2.1). These are very special rectangles generated by the given Markov partition. It would be of natural interest to extend our results to other classes of rectangles, which we do next.

Recall that a rectangle R is a subset of X of a small diameter such that for any points $x, y \in R$ the intersection $W_x^s \cap W_y^u$ of the local stable manifold W_x^s through x and the

[2]Note that in this case the result of Theorem 2.1 can be derived from a recent manuscript by Conze and Raugi [4].

local unstable manifold W_y^u through y is a point that also belongs in R. For $x \in R$ put $W_x^{u,s}(R) = W_x^{u,s} \cap R$. For $x, y, \in R$ put $[x,y] = W_x^s \cap W_y^u$. Then for any $z \in R$ we have

$$R = [W_z^u(R), W_z^s(R)] = \{[x,y] : \, x \in W_z^u(R), y \in W_z^s(R)\} \, .$$

So, R has a direct product structure and $W_z^u(R)$, $W_z^s(R)$ can be thought of as coordinate planes in R. Note that $\partial R = \partial^u R \cup \partial^s R$, where

$$\partial^u R = [W_z^u(R), \partial W_z^s(R)] \qquad \text{and} \qquad \partial^s R = [\partial W_z^u(R), W_z^s(R)]$$

(these sets do not depend on $z \in R$).

We will consider small enough rectangles such that all local unstable manifolds $W_x^u(R)$, $x \in R$ are almost parallel, and so are all stable manifolds $W_x^s(R)$, $x \in R$. Hence, the diameters of our rectangles are $\leq \varepsilon_1$ with some fixed small $\varepsilon_1 > 0$. Our rectangles are not necessarily connected.

Our main assumption must be some sort of 'roundness' of rectangles, the necessity of which we explained above. For any $\varepsilon > 0$ put

$$W_z^u(R, \varepsilon) := \{x \in W_z^u(R) : \, \mathrm{dist}(x, \partial W_z^u(R)) < \varepsilon\}$$

and

$$R_z^u(\varepsilon) := [W_z^u(R, \varepsilon), W_z^s] \, . \tag{2.3}$$

This is a sort of ε-neighborhood of the stable boundary $\partial^s R$. Similarly, the ε-neighborhood of the unstable boundary $\partial^u R$ is defined, call it $R_z^s(\varepsilon)$.

Now fix another constant $\varepsilon_0 \in (0, \varepsilon_1)$ and some constants $C_0 > 0$, $\gamma > 0$.

Definition. We say that a rectangle R is *u-quasiround* if for some $z \in R$
(i) the set $W_z^u(R)$ has (external) diameter $\leq \varepsilon_1$ and internal diameter $\geq \varepsilon_0$ (note that this set will be perfectly round if $\varepsilon_0 = \varepsilon_1$);
(ii) For all $\varepsilon > 0$

$$\mu(R_z^u(\varepsilon)) \leq C_0 |\ln \varepsilon|^{-1-\gamma} \mu(R) \tag{2.4}$$

Similarly, *s-quasiround* rectangles are defined.

Note that the definition of u- and s-quasiroundness depends on the pre-fixed constants $\varepsilon_1, \varepsilon_0, C_0, \gamma$.

The choice of z in this definition is not important, since the same properties will also holds for all $z \in R$, with possibly slightly different values of $\varepsilon_1, \varepsilon_0$ and C_0. The exact values of $\varepsilon_1, \varepsilon_0, C_0, \gamma$ may affect some constants in our estimates, but otherwise will be irrelevant.

Note that if the set $\partial W_z^u(R)$ is smooth or piecewise smooth and the measure on W_z^u induced by μ is smooth, then $\mu(R_z^u(\varepsilon)) \leq \mathrm{const} \cdot \varepsilon \mu(R)$. It is quite common in hyperbolic dynamics to assume that the measure of ε-neighborhoods of boundaries or singularities is bounded by const$\cdot\varepsilon^a$ for some $a > 0$. Our bound (2.4) is milder than that.

Next, we need to consider arbitrary small rectangles that satisfy some sort of roundness condition.

Definition. We call a rectangle R *eventually quasiround* (EQR) if there are two integers $k^- \leq k^+$ such that $T^{k^+}(R)$ is u-quasiround and $T^{k^-}(R)$ is s-quasiround.

The integers $k^\pm$ may not be uniquely defined for a rectangle R, but each of them is defined by R up to a small additive depending on the ratio $\varepsilon_1/\varepsilon_0$, so the choice of $k^\pm$ for a given R will not be important.

EQR rectangles in the Anosov setting play a role similar to that of cylinders for TMC's, and the numbers k^-, k^+ correspond to the endpoints of cylinders. Note, however, that EQR rectangles are not generated by any Markov partitions. On the other hand, we impose the regularity condition (2.4) on the boundary of EQR rectangles, while no such condition was assumed for cylinders.

Note that if $\dim X = 2$, then stable and unstable manifolds are one-dimensional, and, with appropriate choice of ε_0, ε_1, every connected rectangle is EQR. Indeed, the property (i) follows from the uniform hyperbolicity of T and the compactness of X, while the property (ii) follows from our Lemma 4.8 in Section 4 (note that the set $R_z^u(\varepsilon)$ in this case consists of two connected rectangles).

Definition. We say that two EQR rectangles R_1, R_2 with the corresponding integers k_1^-, k_1^+ and k_2^-, k_2^+ characterizing their quasiroundness are *D-nested* for $D \geq 0$ if either $[k_1^-, k_1^+] \subset [k_2^- - D, k_2^+ + D]$ or $[k_2^-, k_2^+] \subset [k_1^- - D, k_1^+ + D]$.

Theorem 2.4 *Let $T : X \mapsto X$ be an Anosov diffeomorphism with a Gibbs measure μ defined by a Hölder continuous potential φ on X, and $D \geq 0$ a constant. Let $\{R_n\}$ be a sequence of EQR rectangles. Assume that for all $m, n \geq 1$ the rectangles R_m, R_n are D-nested. Then $\{R_n\}$ satisfies* (SP) *and hence, if in addition $\sum \mu(R_n) = \infty$, it is an sBC sequence and* (1.1) *holds.*

Examples.
3. If a sequence of EQR rectangles R_n satisfies the condition

$$|k_n^- + k_n^+| \leq D = \text{const} \tag{2.5}$$

then it is an sBC sequence and verifies (1.1).
4. In particular, if T is a linear 2-D toral automorphism and μ the Lebesgue measure, then any sequence of connected rectangles with uniformly bounded ratio of stable and unstable sides (which is sometimes called 'aspect ratio') satisfies the condition (2.5) and hence the conclusion of Theorem 2.4 holds.
5. Let $T : X \mapsto X$ be the baker's transformation of the unit square $X = [0,1] \times [0,1]$ and μ the Lebesgue measure. Note that T is discontinuous but still admits a finite Markov partition. Then any sequence of balls with diverging measures is a BC sequence. Indeed, in each ball $B \subset X$ one can find a 'dyadic' square $R \subset B$ such that $\mu(R) \geq 0.1\mu(B)$. Dyadic squares correspond to 0-centered cylinders in the symbolic space, so one can apply Theorem 2.1 and obtain the sBC property for the dyadic squares, which implies (at least) the BC property for the original balls.

Next, we generalize Example 4 to nonlinear Anosov diffeomorphisms. Let $T: X \mapsto X$, $\dim X = 2$, be an Anosov diffeomorphism of a surface. Recall that in this case every connected rectangle $R \subset X$ is EQR. For a connected rectangle R we denote

$$d^u(R) = \sup_{z\in R} |W^u_z(R)| \qquad \text{and} \qquad d^s(R) = \sup_{z\in R} |W^s_z(R)|\,,$$

where $|W^u|$, $|W^s|$ stand for the Lebesgue measures (lengths) of the corresponding curves W^u, W^s. Let $B \geq 1$. We say that a rectangle R has a *B-bounded aspect ratio* if

$$B^{-1} \leq d^u(R)/d^s(R) \leq B\,.$$

Note that rectangles with B-bounded aspect ratio are, in the geometric sense, close to squares (i.e., 'round'). This geometric version of roundness is somewhat more preferable and easier to check than the dynamical roundness assumed by (2.5).

Theorem 2.5 *Let $T: X \mapsto X$, $\dim X = 2$, be an Anosov diffeomorphism with a Gibbs measure μ defined by a Hölder continuous potential φ on X, and $B \geq 1$ a constant. Let $\{R_n\}$ be a sequence of connected rectangles with (uniformly) B-bounded aspect ratio. Then $\{R_n\}$ satisfies* (SP) *and hence, if in addition $\sum \mu(R_n) = \infty$, it is an sBC sequence and* (1.1) *holds.*

The extensions of Theorems 2.2 and 2.3 to EQR rectangles can also be obtained but are hardly worth pursuing, because the examples of cylinders constructed in 2.2 and 2.3 can be simply projected on X and produce the corresponding examples of rectangles.

3 Proofs for topological Markov chains

The following facts about Gibbs measures are standard:
Fact 1 For any cylinder C defined on an interval Λ

$$c_1\theta_1^{|\Lambda|} \leq \mu(C) \leq c_2\theta_2^{|\Lambda|}\,,$$

where $c_1, c_2 > 0$ and $\theta_1, \theta_2 \in (0,1)$ only depend on the Gibbs measure μ.
Fact 2 Let $C_1 \subset C$ be cylinders defined on intervals Λ_1, Λ (note that in this case $\Lambda_1 \supset \Lambda$), then

$$c_1\theta_1^{|\Lambda_1|-|\Lambda|} \leq \mu(C_1)/\mu(C) \leq c_2\theta_2^{|\Lambda_1|-|\Lambda|}\,.$$

Fact 3 Let C_1, C_2 be cylinders defined on disjoint intervals $[n_1^-, n_1^+]$ and $[n_2^-, n_2^+]$ in $\mathbb{Z}$. Assume, without loss of generality that $n_1^+ < n_2^-$. Then

$$|\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| \leq c_3\theta_3^{n_2^- - n_1^+}\mu(C_1)\mu(C_2)\,,$$

where $c_3 > 0$ and $\theta_3 \in (0,1)$ only depend on the Gibbs measure μ.

Facts 1 and 2 can be proved with the help of a normalized potential for the Gibbs measure μ, see [3]. Fact 3 is proved by R. Bowen in [2].

Let us introduce the following notation. If $\Lambda_1 = [n_1^-, n_1^+]$ and $\Lambda_2 = [n_2^-, n_2^+]$ are two intervals (not necessarily disjoint), define an "asymmetric distance" $\delta(\Lambda_1, \Lambda_2)$ by

$$\delta(\Lambda_1, \Lambda_2) = \min\{D : \Lambda_2 \text{ is in the } D\text{−neighborhood of } \Lambda_1\}$$

Equivalently, $\delta(\Lambda_1, \Lambda_2) = \max\{n_2^+ - n_1^+, n_1^- - n_2^-, 0\}$. Clearly, $\delta(\Lambda_1, \Lambda_2) = 0$ if and only if $\Lambda_2 \subset \Lambda_1$. It is also clear that Λ_1, Λ_2 are D-nested if and only if one of the distances $\delta(\Lambda_1, \Lambda_2)$ and $\delta(\Lambda_2, \Lambda_1)$ does not exceed D.

Lemma 3.1 *If C_1, C_2 are cylinders defined on intervals Λ_1 and Λ_2, respectively, then*

$$|\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| \leq c_4 \theta_4^{\delta(\Lambda_1,\Lambda_2)} \mu(C_1)\,,$$

where $c_4 > 0$ and $\theta_4 \in (0, 1)$ only depend on the Gibbs measure μ.

Proof. This follows from Facts 1 and 3 if Λ_1 and Λ_2 are disjoint, and from Facts 1 and 2 if they are not. □

Proof of Theorem 2.1. We estimate the quantity $R_{mn} = \mu(C_m \cap \sigma^{m-n} C_n) - \mu(C_n)\mu(C_m)$. Without loss of generality, assume that the interval Λ_m is "nested" in Λ_n, i.e. Λ_m lies in the D-neighborhood of Λ_n. Note that we do not assume any relation between m and n, or between $\mu(C_m)$ and $\mu(C_n)$. Our assumption easily implies that $\Lambda_n - (m - n)$ is not in the $(|m-n| - D)$-neighborhood of Λ_m. Applying Lemma 3.1 to the cylinders C_m and $\sigma^{m-n} C_n$, one gets

$$|R_{mn}| \leq c_4 \theta_4^{\delta(\Lambda_m, \Lambda_n - (m-n))} \mu(C_m) \leq c_4 \theta_4^{|m-n|-D} \mu(C_m)$$

Summing up over all n satisfying our nesting condition (that Λ_m is "nested" in Λ_n) gives a quantity bounded by const$\cdot\mu(C_m)$. Now summing up over $m = M, \ldots, N$ proves (SP). □

In the following proofs of Theorems 2.2 and 2.3 we use a special construction. Let T be a measure preserving transformation (invertible or not) of a probability space (X, μ), and let $\{\tilde{A}_k\}$ be a sequence of measurable subsets of X and $\{l_k\}$ a sequence of natural numbers. Put $s_0 = 0$ and $s_k = l_1 + \cdots + l_k$ for $k \geq 1$. Consider a new sequence of sets $\{A_n\}$ defined as follows:

$$T^{1-l_1}\tilde{A}_1, T^{2-l_1}\tilde{A}_1, \ldots, T^{-1}\tilde{A}_1, \tilde{A}_1, T^{1-l_2}\tilde{A}_2, \ldots, T^{-1}\tilde{A}_2, \tilde{A}_2, T^{1-l_3}\tilde{A}_3, \ldots, T^{-1}\tilde{A}_3, \tilde{A}_3, \ldots$$

Note that the nth set in this sequence is

$$A_n = T^{n-s_k}\tilde{A}_k\,, \tag{3.1}$$

where k is defined by $s_{k-1} < n \leq s_k$. We denote this k by by $k = k_n$. We will say that the new sequence, $\{A_n\}$, is *derived* from $\{\tilde{A}_k\}$ and $\{l_k\}$.

Proof of Theorem 2.2. Without loss of generality, assume that l_n is monotonic, $1 \leq l_1 \leq l_2 \leq \cdots$. Let $\{\tilde{C}\}$ be a cylinder defined on some interval $[0, l]$ (alternatively, we can assume

that its center is at zero). Now consider the sequence of cylinders $\{C_n\}$ derived from the constant sequence $\tilde{C}_k = \tilde{C}$ and $\{l_k\}$. Then C_n is defined on an interval Λ_n whose left endpoint lies in the interval $[0, l_k]$ where $k = k_n$ is defined above. Since $\{l_k\}$ is monotonic, the left endpoint of Λ_n lies in $[0, l_n]$, so the nesting condition of Theorem 2.2 is satisfied. It is now easy to see that for $N = l_1 + \cdots + l_k$ we have $E_N = \sum_{n=1}^{N} \mu(C_n) = (l_1 + \cdots + l_k)\mu(\tilde{C})$, while

$$\sum_{m,n=1}^{N} R_{mn} \geq \frac{1}{2}\left(l_1^2 + \cdots + l_k^2\right)\mu(\tilde{C})\,.$$

It is clear that the right hand side of this inequality grows faster than CE_N for any $C > 0$, which violates (SP). □

We write $a_n \approx b_n$ for two sequences of numbers $\{a_n\}$ and $\{b_n\}$ if there are constants $0 < c_1 < c_2 < \infty$ such that $c_1 < a_n/b_n < c_2$ for all n (the constants c_1, c_2 may depend on the topological Markov chain $(\Sigma_{\mathbf{A}}, \sigma)$ and the Gibbs measure μ).

Proof of Theorem 2.3. Let $\{\tilde{C}_k\}$ be a sequence of cylinders defined on intervals $\tilde{\Lambda}_k$ with left endpoints at zero such that $\mu(\tilde{C}_k) \approx 1/(k \ln^2 k)$. (Again, we could assume that the centers of $\{\tilde{\Lambda}_k\}$ are at zero.) It follows from Fact 1 that $|\tilde{\Lambda}_k| \approx \log k$. For each $k \geq 1$, let $l_k = [\varepsilon|\tilde{\Lambda}_k|]$. Consider the sequence of cylinders $\{C_n\}$ derived from $\{\tilde{C}_k\}$ and $\{l_k\}$. Then C_n is defined on an interval Λ_n whose left endpoint lies in the interval $[0, \varepsilon|\Lambda_n|]$. Since $l_k \approx \log k$, we have $\sum \mu(C_n) = \sum_k l_k \mu(\tilde{C}_k) = \infty$. On the other hand, $\sum_k \mu(\tilde{C}_k) < \infty$. Hence, by Lemma 1.2 (i), for a.e. $\underline{\omega} \in \Sigma$ there are at most finitely many $k \geq 0$ such that $\sigma^{s_k}\underline{\omega} \in \tilde{C}_k$. Now, by (3.1), $C_n = \sigma^{n-s_k}\tilde{C}_k$, hence there are at most finitely many n such that $\sigma^n \underline{\omega} \in C_n$. □

Lastly, we give proofs of two propositions from the introduction.

Proof of Proposition 1. Part (i) easily follows from the ergodic theorem. For part (ii), let T be weakly mixing and $\{A_n\}$ contain finitely many distinct subsets of X of positive measure, call them $F_1, \ldots, F_k$. Since $c_1 < E_N/N < c_2$ for some constants $0 < c_1 < c_2 < \infty$, to show that

$$\mu(S_N/E_N - 1)^2 \to 0 \qquad \text{as} \quad N \to \infty \tag{3.2}$$

it is enough to prove that

$$\mu(S_N - E_N)^2 = \sum_{m,n=1}^{N} R_{mn} = o(N^2) \tag{3.3}$$

The weak mixing of T implies that for any F_i, F_j

$$\sum_{n=1}^{N} |\mu(T^{-n}F_i \cap F_j) - \mu(F_i)\mu(F_j)| = o(N)\,,$$

and since we only have finitely many pairs (F_i, F_j), the term $o(N)$ here is uniform in i, j. This completes the proof of (3.3). On the other hand, if (3.2) holds, one can choose a subsequence $\{N_k\}$ such that $S_{N_k}/E_{N_k} \to 1$ almost surely. Thus $S_{N_k} \to \infty$ on a set of full measure, which clearly implies that $\{A_n\}$ is a BC sequence.

Assume now that T is not weakly mixing. If it is not ergodic, the constant sequence $A_n = A$, where A is a nontrivial invariant set, is clearly not a BC sequence. Otherwise T has a factor isomorphic to a rotation of a circle (because T has a non-constant eigenfunction with eigenvalue $\exp(2\pi\theta i)$ with some $0 < \theta < 1$, see e.g. [13], p. 65–68). If θ is rational, then T^k is not ergodic for some k and the claim follows as above. If θ is irrational, then the factor measure is Lebesgue. To finish the proof of (ii) it is then enough to consider an irrational rotation of a circle and find a sequence of (nonempty) arcs $\{A_n\}$ that only contains finitely many distinct arcs but is not BC. This is a simple exercise. Part (iii) follows from the definitions in a straightforward way and is also left as an exercise to the reader. □

Proof of Proposition 1.6. If for some ε there are no measurable subsets A of X with $0 < \mu(A) < \varepsilon$, then (assuming μ is nontrivial) T is not weakly mixing, so the claim follows from the previous proposition. Otherwise there exists a sequence $\{\tilde{A}_k\}$ of sets of positive measure such that $\sum_{k=1}^{\infty} \mu(\tilde{A}_k) < \infty$. Define a sequence $\{l_k\}$ of natural numbers by $l_k = [1/\mu(\tilde{A}_k)] + 1$, and let $\{A_n\}$ be a sequence derived from $\{\tilde{A}_k\}$ and $\{l_k\}$. Then clearly $\sum_{n=1}^{\infty} \mu(A_n) = \infty$. On the other hand, we can argue as in the proof of Theorem 2.3 to show that for a.e. $x \in X$ there are at most finitely many n such that $T^n x \in A_n$. □

4 Proofs for Anosov diffeomorphisms

In this section we use an approach based on the shadowing property and specification. Ruelle recently demonstrated the power and elegance of this approach in [12], and we follow his lines.

We recall certain standard facts about transitive Anosov diffeomorphisms. We will denote by Λ finite or infinite intervals of $\mathbb{Z}$. For a finite interval $\Lambda = [n^-, n^+]$, we denote by $|\Lambda| = n^+ - n^- + 1$ the cardinality of Λ. For two disjoint intervals Λ_1, Λ_2 we denote by

$$\mathrm{dist}(\Lambda_1, \Lambda_2) = \min\{|i - j| : i \in \Lambda_1, j \in \Lambda_2\}$$

the length of the gap between them.

Expansiveness. Any Anosov diffeomorphism $T : X \mapsto X$ is *expansive*, i.e. there is a $\delta > 0$ (called *expansivity constant*) such that

$$\forall k \in \mathbb{Z} \;\; d(T^k x, T^k y) < \delta \qquad \Leftrightarrow \qquad x = y\,.$$

In fact, due to the hyperbolicity of T, for some $C > 0$ and $0 < \theta < 1$ one has

$$\forall |k| \le n \;\; d(T^k x, T^k y) < \delta \qquad \Rightarrow \qquad d(x, y) < C\theta^{-n}\,. \tag{4.1}$$

Let $\Lambda \subset \mathbb{Z}$ be an interval of $\mathbb{Z}$, finite or not. Let $\mathbf{x} = (x_k)_{k\in\Lambda} \in X^\Lambda$. Given $\alpha > 0$, we say that $\mathbf{x}$ is an *α-pseudo-orbit* if

$$d(T^k x, x_{k+1}) < \alpha \qquad \text{whenever} \qquad k, k+1 \in \Lambda\,.$$

We say that the orbit of $x \in X$ *β-shadows* $\mathbf{x}$ if

$$d(T^k x, x_k) < \beta \qquad \forall k \in \Lambda\,.$$

Shadowing lemma. For any $\beta > 0$ there is an $\alpha > 0$ such that every α-pseudoorbit is β-shadowed by a true orbit of some $x \in X$.

Note that if $\Lambda = \mathbb{Z}$ and $\beta < \delta/2$, then the true orbit shadowing $\mathbf{x}$ is unique by the expansivity. We fix a $\beta < \delta/2$ and this fixes the corresponding $\alpha > 0$.

Note that if the pseudoorbit is periodic, then it is shadowed by a true periodic orbit with the same period.

Given $\alpha > 0$, there is an integer $K > 0$ such that for every $x, y \in X$ and $n \geq K$ there is a $z \in X$ such that

$$d(z, x) < \alpha \qquad \text{and} \qquad d(T^n z, y) < \alpha\,,$$

which follows from the topological transitivity of T. (Note that our choice of α made above also fixes K.)

Using this remark, we can interpolate (concatenate) several α-pseudoorbits defined on intervals of $\mathbb{Z}$ separated by gaps of lengths $\geq K$ in the following way.

Specification. Let α-pseudoorbits $\mathbf{x}_j$ be defined on disjoint intervals of $\mathbb{Z}$ separated by gaps of length $\geq K$. Then the $\mathbf{x}_j$ are all β-shadowed by one true orbit of some $x \in X$.

One can also find a periodic orbit that β-shadows all $\mathbf{x}_j$, with period $P := i_{\max} - i_{\min} + K$, where $i_{\max}$ and $i_{\min}$ are the maximum and the minimum points of the union of the intervals of $\mathbb{Z}$ on which the pseudoorbits $\mathbf{x}_j$ are defined.

Due to the expansivity, the number of periodic orbits of period P in the above construction is less than some L independent of the lengths of the intervals of $\mathbb{Z}$ where the pseudoorbits are defined. The value of L only depends on the number of these intervals and the lengths of gaps between them. In our further arguments, we will interpolate no more than four pseudoorbits at a time, and the gaps between them will never exceed $2K$, so we just fix the corresponding constant L.

Now, let $g : X \mapsto \mathbb{R}$ be a Hölder continuous function. The bound (4.1) implies the following.

Approximation of sums along orbits. There is a constant $B = B(g)$ such that

$$\forall k \in [p, q] \quad d(T^k x, T^k y) < \delta \qquad \Rightarrow \qquad \left| \sum_{k=p}^{q} g(T^k x) - \sum_{k=p}^{q} g(T^k y) \right| \leq B\,.$$

Furthermore, let the specification property be used to shadow two finite orbits $\{T^k x'\}$, $k \in \Lambda'$, and $\{T^k x''\}$, $k \in \Lambda''$, with

$$K \leq \operatorname{dist}(\Lambda', \Lambda'') \leq 2K\,,$$

by a periodic orbit of z of period

$$P = |\Lambda'| + |\Lambda''| + \operatorname{dist}(\Lambda', \Lambda'') + K\,,$$

then

$$\left|\sum_{k\in\Lambda'} g(T^k x') + \sum_{k\in\Lambda''} g(T^k x'') - \sum_{k=1}^{P} g(T^k z)\right| \le B' := 2B + 3K||g||_\infty\,. \tag{4.2}$$

Note that B' is a constant, just like B, independent of the lengths of the intervals Λ', Λ''.

For $n \ge 1$, let

$$\mathrm{Fix}(T^n, X) = \{x \in X : T^n x = x\}$$

be the set of periodic points of period n in X.

Periodic orbit approximation of Gibbs measures. Let μ be a Gibbs measure corresponding to a Hölder continuous potential $\varphi : X \mapsto \mathbb{R}$. For each $n \ge 1$, let μ_n be an atomic probability measure concentrated on $\mathrm{Fix}(T^n, X)$ that assigns weight

$$\mu_n(x) = Z_n^{-1} \exp[\varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x)] \tag{4.3}$$

to each point $x \in \mathrm{Fix}(T^n, X)$ (here Z_n is a normalizing factor). Then μ_n weakly converges to μ as $n \to \infty$.

Variational principle. Let $\varphi : X \mapsto \mathbb{R}$ be a continuous function and P_φ its topological pressure. Then

$$\sup_\nu [h_\nu(T) + \nu(\varphi)] = P_\varphi\,, \tag{4.4}$$

where the supremum is taken over all T-invariant probability measures ν on X, and $h_\nu(T)$ is the Kolmogorov-Sinai entropy of ν. Any measure ν that turns (4.4) into an equality is called an equilibrium state for φ. Equilibrium states exist for every continuous function φ. If φ is Hölder continuous on X, the equilibrium state is unique and coincides with the Gibbs measure for the potential φ.

We now prove a few technical lemmas. Let μ be a Gibbs measure on X corresponding to a Hölder continuous potential φ.

We generalize our notation of Section 3 by writing for any two variable quantities A and B

$$A \approx B \qquad \Leftrightarrow \qquad 0 < c_1 < A/B < c_2 < \infty$$

for some constants c_1, c_2 that only depend on $T : X \mapsto X$ and the Gibbs measure μ.

Lemma 4.1 *The normalizing factor (the analogue of partition function) Z_n in (4.3) satisfies*

$$Z_n \approx e^{P_\varphi n}\,.$$

Note that it is standard to compute the topological pressure as

$$P_\varphi = \lim_{n\to\infty} \frac{1}{n} \ln Z_n\,.$$

The estimate in our lemma is sharper than this standard formula.

We need an elementary sublemma that is a modification of a standard one, see Lemma 1.18 in [2].

Sublemma 4.2 *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $|a_{m+n} - a_m - a_n| \le R$ for all $m, n \ge 1$ and some constant $R > 0$. Then $P := \lim_{n\to\infty} a_n/n$ exists. Furthermore, $|a_n - Pn| \le 2R$ for all n.*

Proof. Fix an $m \ge 1$. For $n \ge 1$, write $n = km + l$ with $0 \le l \le m-1$. Then it follows by induction on k that $|a_n - ka_m - a_l| \le kR$. Hence,

$$\left|\frac{a_n}{n} - \frac{ka_m}{km+l} - \frac{a_l}{km+l}\right| \le \frac{kR}{km+l}\,.$$

Letting $n \to \infty$ gives

$$\frac{a_m}{m} - \frac{R}{m} \le \liminf_n \frac{a_n}{n} \le \limsup_n \frac{a_n}{n} \le \frac{a_m}{m} + \frac{R}{m}\,.$$

Hence, $P := \lim a_n/n$ exists. Next, assume that $a_m > Pm + 2R$ for some m. Then $a_{2^n m} > 2^n mP + (2^n+1)R$ which follows by induction on n. Hence $\limsup a_n/n \ge P + R/m$, a contradiction. A similar contradiction results from the assumption $a_m < Pm - 2R$. □

Proof of Lemma 4.1. It is enough to show that

$$R := \sup_{m,n} |\ln Z_{m+n} - \ln Z_m - \ln Z_n| < \infty$$

and apply the previous sublemma to the sequence $a_n = \ln Z_n$. So, we need to show that

$$Z_{m+n} \approx Z_m Z_n\,.$$

For fixed n, m, put $\Lambda' = [0, m-K]$ and $\Lambda'' = [m, m+n-K]$. For any $x \in \mathrm{Fix}(T^{m+n}, X)$ consider $\mathbf{x}' = \{x, \dots, T^{m-K}x\}$ and $\mathbf{x}'' = \{T^m x, \dots, T^{m+n-K}x\}$, these are two pseudoorbits defined on the intervals Λ' and Λ'' separated by a gap of length K. Each of them can be shadowed by a true periodic orbit, of periods m and n, respectively, and there are at most L of those periodic orbits for each of $\mathbf{x}'$ and $\mathbf{x}''$. On the other hand, for every pair of periodic orbits $y' \in \mathrm{Fix}(T^m, X)$ and $y'' \in \mathrm{Fix}(T^n, X)$ consider two pseudoorbits $\mathbf{y}' = \{y', \dots, T^{m-K}y'\}$ defined on Λ' and $\mathbf{y}'' = \{y'', \dots, T^{n-K}y'\}$ defined on the interval Λ'' by associating $T^i y''$ to $m + i \in \Lambda''$. Then there is a true periodic orbit of period $m+n$ shadowing both $\mathbf{y}'$ and $\mathbf{y}''$, and the number of those periodic orbits does not exceed L. Now the result follows from (4.2). □

Note that the potential $\varphi - P_\varphi$ corresponds to the same measure μ and has zero topological pressure. Hence we may just assume that $P_\varphi = 0$ in what follows. Then $Z_n \approx 1$.

We assume, as we may, that ε_1 in the definition of EQR rectangles does not exceed the expansivity constant δ.

Lemma 4.3 *Let R be an EQR rectangle with integers k^+ and k^- characterizing the quasiroundedness of R. Let $x \in R$. Then*

$$\mu(R) \approx \exp[\varphi(T^{k^-}x) + \cdots + \varphi(T^{k^+}x)]\,.$$

Proof. Consider a pseudoorbit $\mathbf{x} = \{T^{-k^-}x, \ldots, T^{k^+}x\}$ defined on the interval $\Lambda = [-k^-, k^+]$. Let $n \gg |\Lambda|$. Note that $y \in R$ if and only if (i) $W^s(T^{k^+}y)$ intersects $W^u(T^{k^+}x)$, and (ii) $W^u(T^{k^-}y)$ intersects $W^s(T^{k^-}x)$. This definitely happens if the orbit $\mathbf{y} = \{T^{k^-}y, \ldots, T^{k^+}y\}$ ε_0-shadows $\mathbf{x}$. On the other hand, if $y \in R$, then $\mathbf{y}$ ε_1-shadows $\mathbf{x}$. Hence, we can apply our previous estimates with $\alpha = \varepsilon_0$ and $\alpha = \varepsilon_1$, the value of α only affects the values of all constants, which are not essential. So, we may simply assume that $y \in R$ if and only if $\mathbf{y}$ α-shadows $\mathbf{x}$.

Now, for any $y \in \mathrm{Fix}(T^n, X)$ consider the pseudoorbit $\mathbf{y} = \{T^{k^+ + K}y, \ldots, T^{n+k^- - K}y\}$ defined on the interval $\Lambda' = [k^+ + K, n + k^- - K]$. Then $\mathbf{y}$ is shadowed by a true periodic orbit of period $p := n - (k^+ - k^-) - K + 1$, and the number of those periodic orbits is less than L. On the other hand, for any $z \in \mathrm{Fix}(T^p, X)$ consider a pseudoorbit $\mathbf{z} = \{z, \ldots, T^{p-1}z\}$ defined on the interval Λ' by associating $T^i z$ to $k^+ + K + i \in \Lambda'$. Then there is a true periodic orbit of period n shadowing both $\mathbf{x}$ and $\mathbf{z}$, and the number of those periodic orbits does not exceed L. Now the result follows from (4.2) and the facts $Z_n \approx 1$ and $Z_p \approx 1$. $\square$

Lemma 4.4 *Let R_1, R_2 be two EQR rectangles with integers $k_1^\pm$ and $k_2^\pm$ characterizing the quasiroundedness of R_1, R_2. Denote $\Lambda_i = [k_i^-, k_i^+]$ for $i = 1, 2$. Let $x \in R_1 \cap R_2$. Then*

$$\mu(R_1 \cap R_2) \le c \cdot \exp\left[\sum_{i \in \Lambda_1 \cup \Lambda_2} \varphi(T^i x)\right],$$

with some $c > 0$ that only depends on the Gibbs measure μ.

Proof. The proof of the previous lemma applies with the following simple adjustments. Note that if $y \in R_1 \cap R_2$, then the orbit of y ε_1-shadows that of x on $\Lambda_1 \cup \Lambda_2$. So, to get an upper bound on $\mu(R_1 \cap R_2)$, we can take into account all n-periodic orbits that α-shadow the orbit of x on $\Lambda_1 \cup \Lambda_2$ with $\alpha = \varepsilon_1$. Now, if Λ_1 and Λ_2 overlap, the argument is exactly like in the proof of the previous lemma. Let Λ_1 and Λ_2 be disjoint with $\mathrm{dist}(\Lambda_1, \Lambda_2) = J$. If $J \le 2K$, we can simply disregard such a small gap and apply the previous argument. If $J > 2K$, we replace the part of the orbit of $y \in \mathrm{Fix}(T^n, X)$ of length J between Λ_1 and Λ_2 by periodic orbits of period $J - 2K$. To conclude the argument, we now need an obvious extension of (4.2) from two to four pseudoorbits with gaps of length K in between. This extension is straightforward. $\square$

Lemma 4.5 *There is a constant $\Delta > 0$ such that for all $n \ge 1$ and $x \in \mathrm{Fix}(T^n, X)$ we have*

$$\varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x) \le -\Delta n\,.$$

Proof. Let δ_x be the delta measure concentrated at x. The measure

$$\delta_{x,n} = \frac{1}{n}(\delta_x + \cdots + \delta_{T^{n-1}x})$$

is T-invariant, so by the variational principle we have

$$\delta_{x,n}(\varphi) = \frac{1}{n}\Big(\varphi(x) + \cdots + \varphi(T^{n-1}x)\Big) \le 0\,.$$

We now need to prove that

$$\sup_{n\ge 1} \sup_{x\in \mathrm{Fix}(T^n,X)} \delta_{x,n}(\varphi) < 0\,.$$

If this is not true, then there is a sequence of periodic points $x_k \in \mathrm{Fix}(T^{n_k}, X)$ such that $\delta_{x_k,n_k}(\varphi) \to 0$. We take any limit point of the sequence of measures δ_{x_k,n_k} in the weak topology, it will be a T-invariant measure, call it ν. We have $\nu(\varphi) = 0$, so by the uniqueness part of the variational principle $\nu = \mu$, so $\mu(\varphi) = 0$ and hence $h_\mu(T) = 0$. But it is known that $h_\mu(T) > 0$ for any Gibbs measure, a contradiction. □

Combining this lemma with the specification property and (4.2) gives

Corollary 4.6 *There is a constant $B_0 > 0$ such that for all $n \ge 1$ and $x \in X$*

$$-\Delta_0 n \le \varphi(x) + \varphi(Tx) + \cdots + \varphi(T^{n-1}x) \le B_0 - \Delta n$$

with $\Delta_0 = ||\varphi||_\infty$.

We can now prove analogues of Facts 1 and 2 of Section 3 for Anosov diffeomorphisms. Our constants, such as c_i, θ_i, will only depend on the Gibbs measure μ and the values of $\varepsilon_0, \varepsilon_1$ in the definition of EQR rectangles. We use notation of Lemmas 4.3 and 4.4.

Lemma 4.7 *Let R be an EQR rectangle and $k := k^+ - k^-$. Then*

$$c_5\theta_5^k \le \mu(R) \le c_6\theta_6^k$$

with some $c_5, c_6 > 0$ and $\theta_5, \theta_6 \in (0,1)$.

Lemma 4.8 *Let R_1, R_2 be EQR rectangles with the corresponding intervals $\Lambda_i = [k_i^-, k_i^+]$, $i = 1, 2$. Put $k := |\Lambda_2 \setminus \Lambda_1|$. Then*

$$\mu(R_1 \cap R_2) \le c_7\theta_7^k \mu(R_1)$$

with some $c_7 > 0$ and $\theta_7 \in (0,1)$.

Proof. Lemmas 4.7 and 4.8 follow from Lemmas 4.3 and 4.4 and Corollary 4.6. □

Note that so far we only used the property (i) of the quasiround rectangles, we did not use (2.4).

Lemma 4.9 *Let R_1, R_2 be EQR rectangles such that the intervals $\Lambda_1 = [k_1^-, k_1^+]$ and $\Lambda_2 = [k_2^-, k_2^+]$ are disjoint. Put $k := \operatorname{dist}(\Lambda_1, \Lambda_2)$. Then*

$$|\mu(R_1 \cap R_2) - \mu(R_1)\mu(R_2)| \le c_8 \frac{\mu(R_1) + \mu(R_2)}{|ak+b|^{1+\gamma}}$$

with some constants $c_8 > 0$, $a > 0$ and b.

Proof. Our proof uses Markov partitions and symbolic dynamics. Let $\mathcal{R}$ be a Markov partition and Σ the corresponding symbolic space, a topological Markov chain. We now partition the rectangles R_1 and R_2 into subrectangles generated by the Markov partition $\mathcal{R}$ as follows. Let $C \subset \Sigma$ be a cylinder defined on an interval $\Lambda \subset \mathbb{Z}$. We say that its projection $\pi(C)$ is *properly inside* R_i, $i = 1, 2$, if

(i) $\pi(C) \subset R_i$, and

(ii) for any larger cylinder $C' \supset C$ its projection $\pi(C')$ is not a subset of R_i.

Denote by $\mathcal{C}_i$ the collection (in general, countable) of cylinders that are properly inside R_i. Since R_i is a rectangle, one can easily check that all the cylinders in $\mathcal{C}_i$ are disjoint. Next, it follows from the assumption (2.4) that $\mu(\partial R_i) = 0$, hence

$$\mu(R_i \setminus \bigcup_{C \in \mathcal{C}_i} \pi(C)) = 0\,,$$

i.e. the rectangles $\pi(C)$, $C \in \mathcal{C}_i$, make a (mod 0) partition of R_i.

Now consider the collection $\mathcal{C}_1$ and an arbitrary cylinder $C \in \mathcal{C}_1$ defined on an interval $\Lambda = [k^-, k^+]$. Observe that if $t := k^+ - k_1^+ > 0$, then, using the notation of (2.3), we have $\pi(C) \subset R^u_{1,z}(\varepsilon)$ with $\varepsilon = c\theta^t$ for any $z \in R_1$. Here $c > 0$ and $\theta \in (0,1)$ are constants determined by the hyperbolicity properties of T and the sizes of rectangles of the Markov partition $\mathcal{R}$. Similarly, if $C \in \mathcal{C}_2$ is defined on an interval $\Lambda = [k^-, k^+]$ and $t = k_1^- - k^- > 0$, then $\pi(C) \subset R^s_{2,z}(\varepsilon)$ with $\varepsilon = c\theta^t$.

Now define subcollections $\mathcal{C}_i' \subset \mathcal{C}_i$ for $i = 1, 2$ that contain all cylinders C defined on intervals $\Lambda = [k^-, k^+]$ satisfying $k^+ - k_1^+ > k/3$ for $i = 1$ and $k_1^- - k^- > k/3$ for $i = 2$ (recall that $k = \operatorname{dist}(\Lambda_1, \Lambda_2)$). By the assumption (2.4)

$$\mu\left(\cup_{C \in \mathcal{C}_i'} \pi(C)\right) \le C_0 \frac{\mu(R_i)}{|ak+b|^{1+\gamma}}$$

with constants $a = -\ln \theta^{1/3} > 0$ and $b = -\ln c$. So, the parts $\pi(C)$, $C \in \mathcal{C}_i'$, can be removed from R_i with no harm. Denote by

$$\tilde{R}_i = R_i \setminus \left(\cup_{C \in \mathcal{C}_i'} \pi(C)\right)$$

the remaining parts of R_i.

Note that $\tilde{R}_1$ and $\tilde{R}_2$ consist (mod 0) of projections of cylinders $C' \in \mathcal{C}_1 \setminus \mathcal{C}_1'$ and $C'' \in \mathcal{C}_2 \setminus \mathcal{C}_2'$, respectively, and the gap between the intervals on which C' and C'' are

defined is always $\geq k/3$. Hence we can use the subadditivity of the correlation function and Fact 3 of Section 3 to get

$$\begin{aligned}
|\mu(\tilde{R}_1 \cap \tilde{R}_2) - \mu(\tilde{R}_1)\mu(\tilde{R}_2)| &= |\mu((\cup \pi(C')) \cap (\cup \pi(C'')) - \mu(\cup \pi(C'))\mu(\cup \pi(C''))| \\
&\leq \sum_{C'} \sum_{C''} |\mu(\pi(C') \cap \pi(C'')) - \mu(\pi(C'))\mu(\pi(C''))| \\
&\leq \sum_{C'} \sum_{C''} c_3 \theta_3^{k/3} \mu(\pi(C'))\mu(\pi(C'')) \\
&\leq c_3 \theta_3^{k/3} \mu(\tilde{R}_1)\mu(\tilde{R}_2)\,.
\end{aligned}$$

This completes the proof of Lemma 4.9. □

Lemma 4.10 *Let R_1, R_2 be EQR rectangles with the corresponding intervals Λ_1 and Λ_2. Then*

$$|\mu(R_1 \cap R_2) - \mu(R_1)\mu(R_2)| \leq c_9 \frac{\mu(R_1) + \mu(R_2)}{|a_1 \delta(\Lambda_1, \Lambda_2) + b|^{1+\gamma}}\,.$$

Here $\delta(\Lambda_1, \Lambda_2)$ is the asymmetric distance defined in Section 3, $a_1 = a/2$, and $c_9 > 0$ is a constant.

Proof. If $|\Lambda_2| \geq \frac{1}{2}\delta(\Lambda_1, \Lambda_2)$, then, by Lemmas 4.7 and 4.8, both $\mu(R_1 \cap R_2)$ and $\mu(R_1)\mu(R_2)$ are bounded from above by $c\theta^{\delta(\Lambda_1,\Lambda_2)}\mu(R_1)$, with some $c > 0$ and $0 < \theta < 1$ depending only on μ. Otherwise $\mathrm{dist}(\Lambda_1, \Lambda_2) \geq \delta(\Lambda_1, \Lambda_2)/2$, and the claim follows from Lemma 4.9. □

Proof of Theorem 2.4 goes by the same lines as the proof of Theorem 2.1. We estimate the quantity $R_{mn} = \mu(R_m \cap T^{m-n}R_n) - \mu(R_n)\mu(R_m)$. Without loss of generality, assume that $\delta(\Lambda_n, \Lambda_m) \leq D$. By Lemma 4.10 (applied to the rectangles R_m and $T^{m-n}R_n$), we have

$$|R_{mn}| \leq c_9 \frac{\mu(R_m) + \mu(R_n)}{|a_1|m-n| - a_1 D + b|^{1+\gamma}}\,.$$

We use this bound if $|m - n| \geq D - b/a_1$, otherwise we can use an obvious bound

$$|R_{mn}| \leq \mu(R_m) + \mu(R_n)\,.$$

Summing up over all over all n with $\delta(\Lambda_n, \Lambda_m) \leq D$, and then over $m = M, \ldots, N$, proves (SP). □

For the proof of Theorem 2.5, we need two more lemmas. Recall that now $\dim X = 2$ and every connected rectangle is EQR.

Lemma 4.11 *Let R_m and R_* be two connected rectangles with B-bounded aspect ratio, and $k_m^\pm$, $k_*^\pm$ integers characterizing their quasiroundness. Assume that $R_m \cap R_* \neq \emptyset$. If $d^u(R_m) \geq d^u(R_*)$, then*

$$k_*^+ \geq k_m^+ + c_{10} \ln d^u(R_m)/d^u(R_*)\,.$$

Similarly, if $d^s(R_m) \geq d^s(R_*)$, then

$$k_*^- \leq k_m^- - c_{10} \ln d^s(R_m)/d^s(R_*)\,.$$

Here $c_{10} = c_{10}(\varepsilon_0, \varepsilon_1, B) > 0$ is a constant.

Proof. This follows from standard distortion bounds. $\square$

Proof of Theorem 2.5. Denote by $k_n^\pm$ the integers characterizing the quasiroundness of R_n. We may assume that all R_n are small enough, and then the uniform boundedness of their aspect ratio ensures that $k_n^+ \geq 0$ and $k_n^- \leq 0$ for all $n \geq 1$.

We estimate the quantity $R_{mn} = \mu(R_m \cap T^{m-n} R_n) - \mu(R_n)\mu(R_m)$. The set $R_* = T^{m-n} R_n$ is a connected rectangle whose quasiroundness is characterized by the integers $k_*^\pm := k_n^\pm + (n-m)$. Without loss of generality, assume that $d^u(R_m) \geq d^u(R_n)$.

We consider three cases:

Case 1. Assume that either (i) $n - m \geq 2k_m^+$ or (ii) $n - m \leq 2k_m^-$. In the case (i) we have

$$k_*^+ - k_m^+ \geq n - m - k_m^+ \geq |n-m|/2\,,$$

and in the case (ii) we have

$$k_m^- - k_*^- \geq k_m^- - (n-m) \geq |n-m|/2\,.$$

In either case we apply Lemma 4.10 and obtain

$$|R_{mn}| \leq c_9 \frac{\mu(R_m) + \mu(R_n)}{|a_2|n-m| + b|^{1+\gamma}}$$

with $a_2 = a_1/2 > 0$.

Case 2. Assume that $2k_m^- \leq n - m \leq 2k_m^+$ and $R_* \cap R_m \neq \emptyset$. If $n > m$, then $d^u(R_*) \leq \theta^{n-m} d^u(R_m)$, and if $n \leq m$, then $d^s(R_*) \leq \theta^{m-n} d^s(R_m)$ for some constant $\theta < 1$, due to the uniform hyperbolicity of T. Hence, Lemma 4.11 implies that if $n > m$, then

$$k_*^+ - k_m^+ \geq c_{11}|n-m|$$

and if $n \leq m$, then

$$k_m^- - k_*^- \geq c_{11}|n-m|$$

with some constant $c_{11} > 0$. Again, we use Lemma 4.10 and obtain

$$|R_{mn}| \leq c_9 \frac{\mu(R_m) + \mu(R_n)}{|a_3|n-m| + b|^{1+\gamma}}$$

with $a_3 = c_{11} a_1$.

Case 3. Assume that $2k_m^- \leq n - m \leq 2k_m^+$ and $R_* \cap R_m = \emptyset$. Then $R_{mn} = \mu(R_m)\mu(R_n)$. It follows from Lemma 4.7 that

$$a_4 |\ln \mu(R_m)| + b_4 \leq k_m^+ - k_m^- \leq a_5 |\ln \mu(R_m)| + b_5$$

with some $a_4, a_5 > 0$ and $-\infty < b_4, b_5 < \infty$, and similar bounds hold for R_n. Our assumption $d^u(R_n) \leq d^u(R_m)$ and the B-boundedness of aspect ratio imply that $k_n^+ \geq \varepsilon_2 k_m^+$ and $k_n^- \leq \varepsilon_2 k_m^-$ for some constant $\varepsilon_2 > 0$, due to uniform bounds on expansion and contraction rates of T. Therefore,

$$\mu(R_n) \leq c_{12} [\mu(R_m)]^\kappa$$

with some constants $c_{12} > 0$ and $\kappa > 0$. Holding m fixed and summing over all n that satisfy the conditions of Case 3 gives

$$\sum_n R_{mn} \le 2c_{12}[\mu(R_m)]^{1+\kappa}(a_5|\ln\mu(R_m)| + b_5) \le c_{13}\mu(R_m)$$

with some constant $c_{13} > 0$.

Lastly, summing up over all $m, n = M, \ldots, N$ proves (SP). $\square$

Acknowledgements. The authors want to thank Vitaly Bergelson, Jean-Pierre Conze, Dmitry Dolgopyat, Yves Guivarc'h, Andres del Junco and Albert Raugi for helpful discussions, and the referee for useful comments.

References

[1] M. Boshernitzan, *Quantitative recurrence results*, Invent. Math. **113** (1993), 617–631.

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes Math. **470**, Springer-Verlag, Berlin, 1975.

[3] N. Chernov, *Invariant measures for hyperbolic dynamical systems*, to appear in In: *Handbook of Dynamical Systems, Vol. I*, Ed. A. Katok and B. Hasselblatt, Elsevier.

[4] J.-P. Conze and A. Raugi, *Convergence des potentiels pour un opérateur de transfert, applications aux systèmes dynamiques et aux chaînes de Markov*, manuscript.

[5] D. Dolgopyat, *Limit theorems for partially hyperbolic systems*, manuscript.

[6] R. Durrett, *Probability: theory and examples*, Wadsworth & Brooks/Cole, 1991

[7] N. Friedman and E. Thomas, *Higher order sweeping out*, Ill. J. Math. **29** (1985), 401–417.

[8] R. Hill and S. Velani, *Ergodic theory of shrinking targets*, Invent. Math. **119** (1995), 175–198.

[9] D. Kleinbock and G. Margulis, *Logarithm laws for flows on homogeneous spaces*, Inv. Math. **138** (1999), 451–494.

[10] U. Krengel, *On the individual ergodic theorem for subsequences*, Ann. of Math. Stat. **42** (1971), 1091-1095.

[11] W. Philipp, *Some metrical theorems in number theory*, Pacific J. Math. **20** (1967), 109–127.

[12] D. Ruelle, *Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics*, J. Statist. Phys. **95** (1999), 393–468.

[13] K. Petersen, *Ergodic theory*, Camb. Univ. Press, 1983.

[14] V. Sprindžuk, *Metric theory of Diophantine approximations*, J. Wiley & Sons, New York-Toronto-London, 1979.